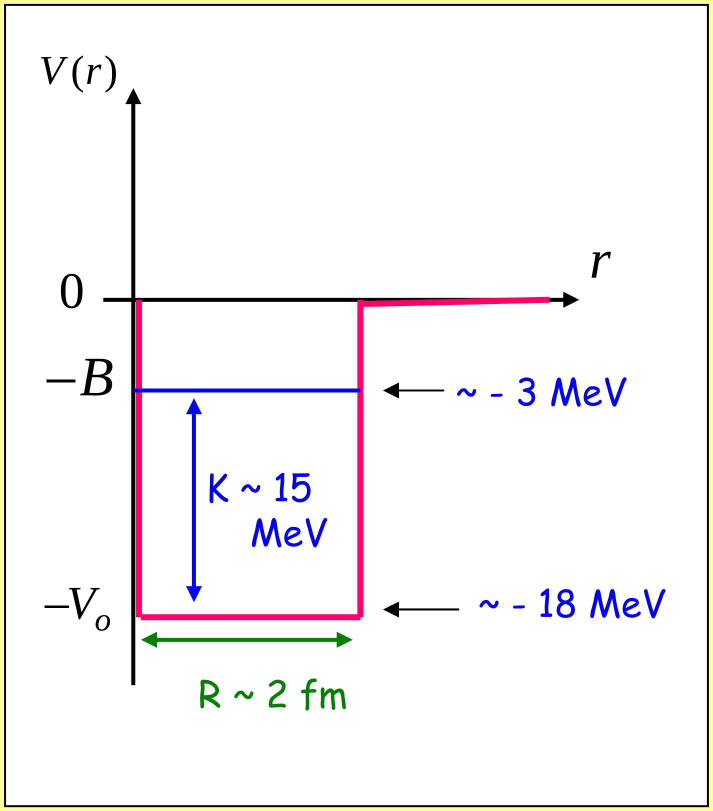
16.451 Lecture 23: A potential model for nuclei

Nov. 28, 2005



From last class:
a simple square well potential V(r)

- We have a complicated system of A nucleons.
- About half of them are protons, so a repulsive (+ve energy) term has to be added to the square well to account for this (~ few MeV)

How to connect this model to something observable?

<u>Independent particle model:</u>

- Assume independent particle motion in some average nuclear potential V(r) as shown.
- Then we can fill the eigenstates of the potential to maximum occupancy to form a nucleus, as is done with electrons in atoms (to 1st order...)

Connection to average nuclear properties:

- The binding energy of each nucleon, in our model, is a few MeV.
- The potential energy of a bound nucleon is **negative**, by ~ 0.3% of its rest mass energy, which therefore has to show up as a decrease in its mass.
- For A nucleons, the **total binding energy** is:

mass of nucleus, M

$$B = \sum_{i=1}^{A} B_i = \sum_{i=1}^{A} m_i - M$$

The average **binding energy per nucleon, B/A**, can be determined from mass data and used to refine a model for $V(r)$; it ranges systematically from about 1 – 9 MeV as a function of mass number for the stable isotopes.

Reference: F&H ch. 16

- By convention, we set the mass of the carbon-12 **atom** as a standard.
- Denote atomic masses with a "script" **M**, measured in **atomic mass units**, ***U***

M (^{12}C) $\equiv$ 12.0000000000 ..... **U** *(exact!)* → 1 **U** = 931.494 MeV *(expt.)*

Calculation for carbon-12:

$$\left.\begin{aligned} m_p &= 938.2 \text{ MeV} \\ m_n &= 939.6 \text{ MeV} \\ m_e &= 0.511 \text{ MeV} \end{aligned}\right\} \qquad \begin{aligned} 6 \times \sum_i m_i &= 11{,}269.8 \text{ MeV} \\ 12\,U &= 11{,}178.0 \text{ MeV} \end{aligned}$$

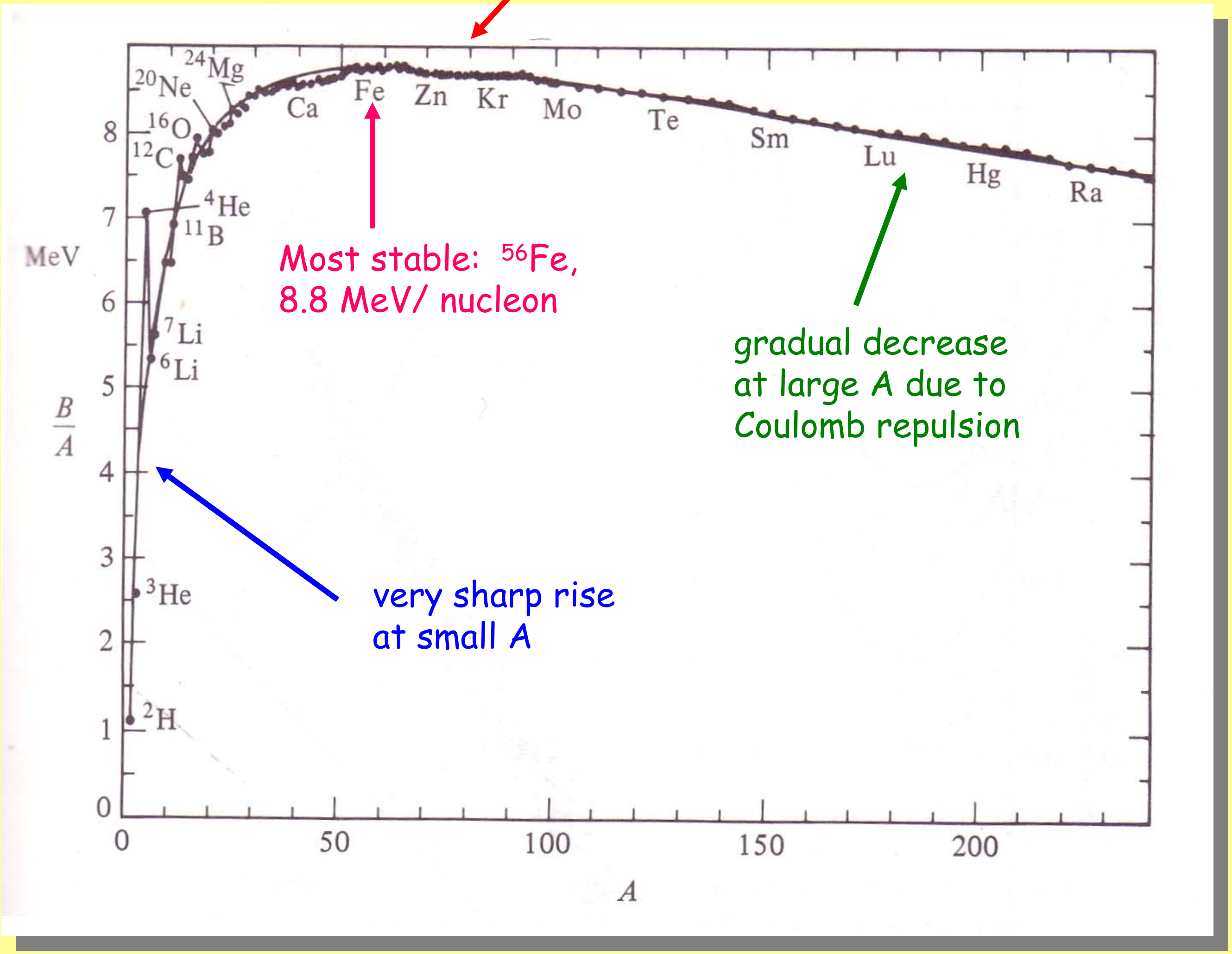
$$\boxed{B\,({}^{12}_{6}C) = \sum_i m_i - M = 91.8 \text{ MeV}}$$

Binding energy per nucleon in ^{12}C: B/A = 7.8 MeV;

Contrast to the deuteron ^{2}H: B/A = 1.1 MeV

The famous Binding Energy per Nucleon curve:

almost flat, apart from Coulomb effects



Implications of the B/A curve:

Greater binding energy implies lower mass, greater stability.

Energy is released when configurations of nucleons change to populate the larger B/A region → nuclear energy generation, e.g.

Fusion reactions at small A release substantial energy **because the B/A curve rises steeply at small A:**

$$\left.\begin{aligned} {}^{2}H + {}^{2}H &\Rightarrow {}^{3}He + n + 3.27\ \mathrm{MeV} \\ &\Rightarrow {}^{3}H + p + 4.03\ \mathrm{MeV} \\ {}^{2}H + {}^{3}H &\Rightarrow {}^{4}He + n + 17.6\ \mathrm{MeV} \end{aligned}\right\}$$

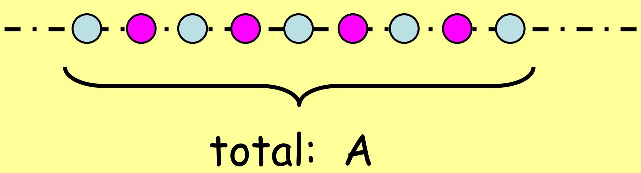
Binding energy of products is greater than the sum of the binding energies of the initial species.

Fission reactions at large A release energy because the products have greater binding energy per nucleon than the initial species:

$$ {}^{235}U + n \Rightarrow \underbrace{{}^{100}X + {}^{133}Y}_{\text{distribution of final products}} + 3n + 200\ \mathrm{MeV} $$

1. Volume and Surface terms:

First consider a 1-dimensional row of nucleons with interaction energy per pair = ε



$$B = \sum_{i=1}^{A} 2\varepsilon - \Delta = 2\varepsilon\, A - \Delta$$

each has 2 neighbors

correction for the ends

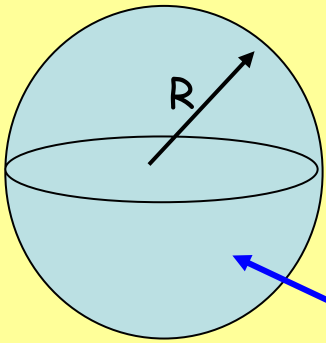
$$\frac{B}{A} = 2\varepsilon - \frac{\Delta}{A}$$

Approximately constant, with end effects relatively smaller at large *A*.

By analogy, for a 3-d nucleus, **there should be both volume and surface terms with the opposite sign**, the surface nucleons having less binding energy:

$$B = a_V A - a_S A^{2/3} \Rightarrow \frac{B}{A} = a_V - a_S A^{-1/3}$$

2. Coulomb term:



total charge + Ze

for a uniform sphere,

$$E_{Coul} = \int \frac{q(r)\,dq}{4\pi\varepsilon_o\, r} = \frac{3}{5}\,\frac{(Ze)^2}{4\pi\varepsilon_o R}$$

This effect increases the total energy and so decreases the binding energy.

Simple model: $\Delta B = -\ a_C\ Z^2\ A^{-1/3}$

But this is not quite right, because in a sense it includes the Coulomb self energy of a single proton by accounting for the integral from 0 to r_p ~ 0.8 fm. The nucleus has fuzzy edges anyway, so we will have to fit the coefficient a_c to mass data.

Solution: let ΔB scale as the number of proton pairs and include a term:

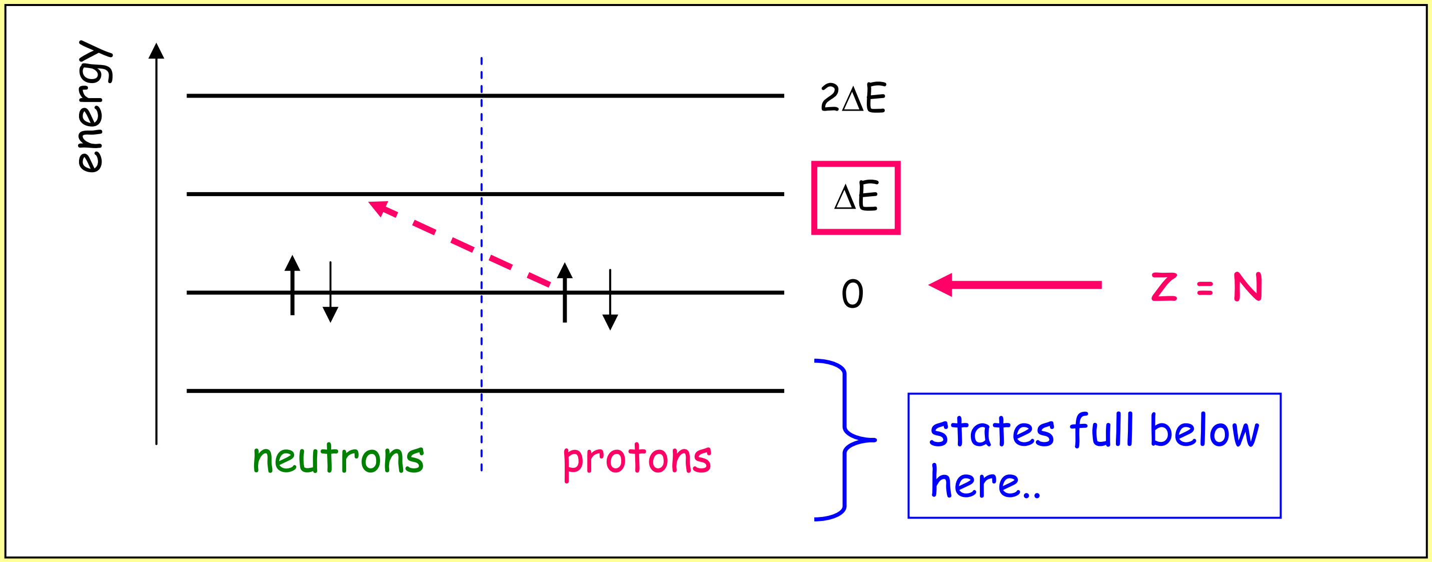
$$\Delta B = -\,a_C\ Z\,(Z-1)\ A^{-1/3} \quad \Rightarrow \quad \frac{\Delta B}{A} = -\ a_C\ Z\,(Z-1)\ A^{-4/3}$$

3. Symmetry Term:

So far, our formula doesn't account for the tendency for light nuclei to have Z = N. The nuclear binding energy ultimately results from filling allowed energy levels in a potential well V(r). **The most efficient way to fill these levels is with Z = N:**

Simplest model: identical nucleons as a Fermi gas, i.e. noninteracting spin- ½ particles in a box. Two can occupy each energy level. The level spacing ~ 1/A.

A mismatch between Z and N costs an energy price of ΔE at fixed A as shown.



$$\Delta B = -a_A\,(Z-N)^2\,A^{-1} = -a_A\,(A-2Z)^2\,A^{-1}$$

4. Pairing Term:

Finally, recall from slide 1 that for the case of **even A**, there are 177 stable nuclei with Z and N both even, and only 6 with Z and N both odd. **Why?**

→ Configurations for which protons and neutrons separately can form pairs must be much more stable. All the even-even cases have $J^{\pi} = 0^{+}$, implying that neutrons and protons have lower energy when paired to total angular momentum zero.

Solution: add an empirical **pairing term** to the binding energy formula:

$$\Delta B_{pair} \equiv \delta = \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix} a_p \, A^{-3/4}$$

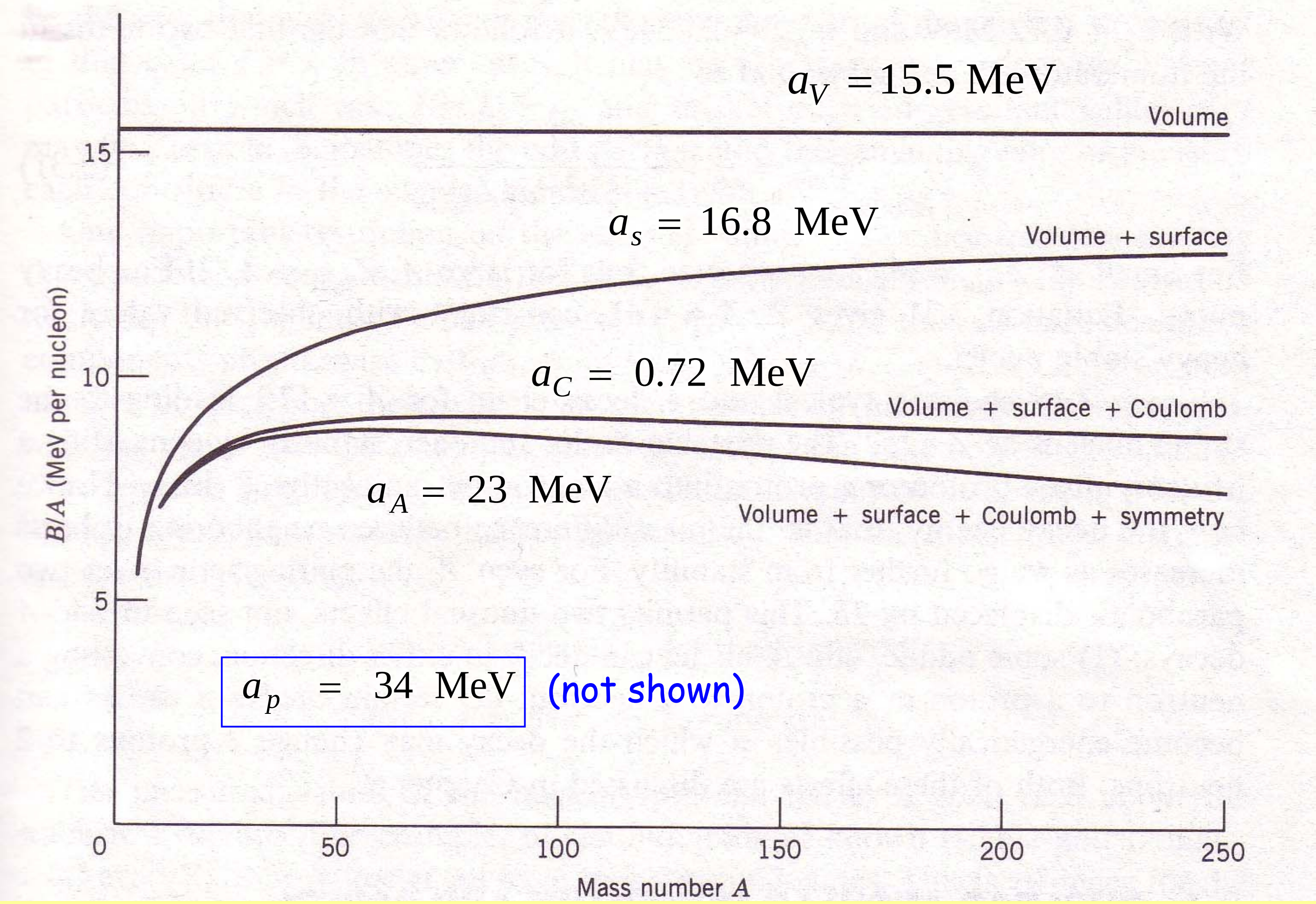
with +1 for even-even, 0 for even-odd, and -1 for odd-odd

Full expression:

$$B(Z,A) = a_V A - a_S A^{2/3} - a_C \, Z\,(Z-1)\, A^{-1/3} - a_A \,(A-2Z)^2 \, A^{-1} + \delta$$

Fitting of coefficients to data:

$$B(Z,A) = a_V A - a_S A^{2/3} - a_C Z(Z-1)A^{-1/3} - a_A (A-2Z)^2 A^{-1} + \delta$$

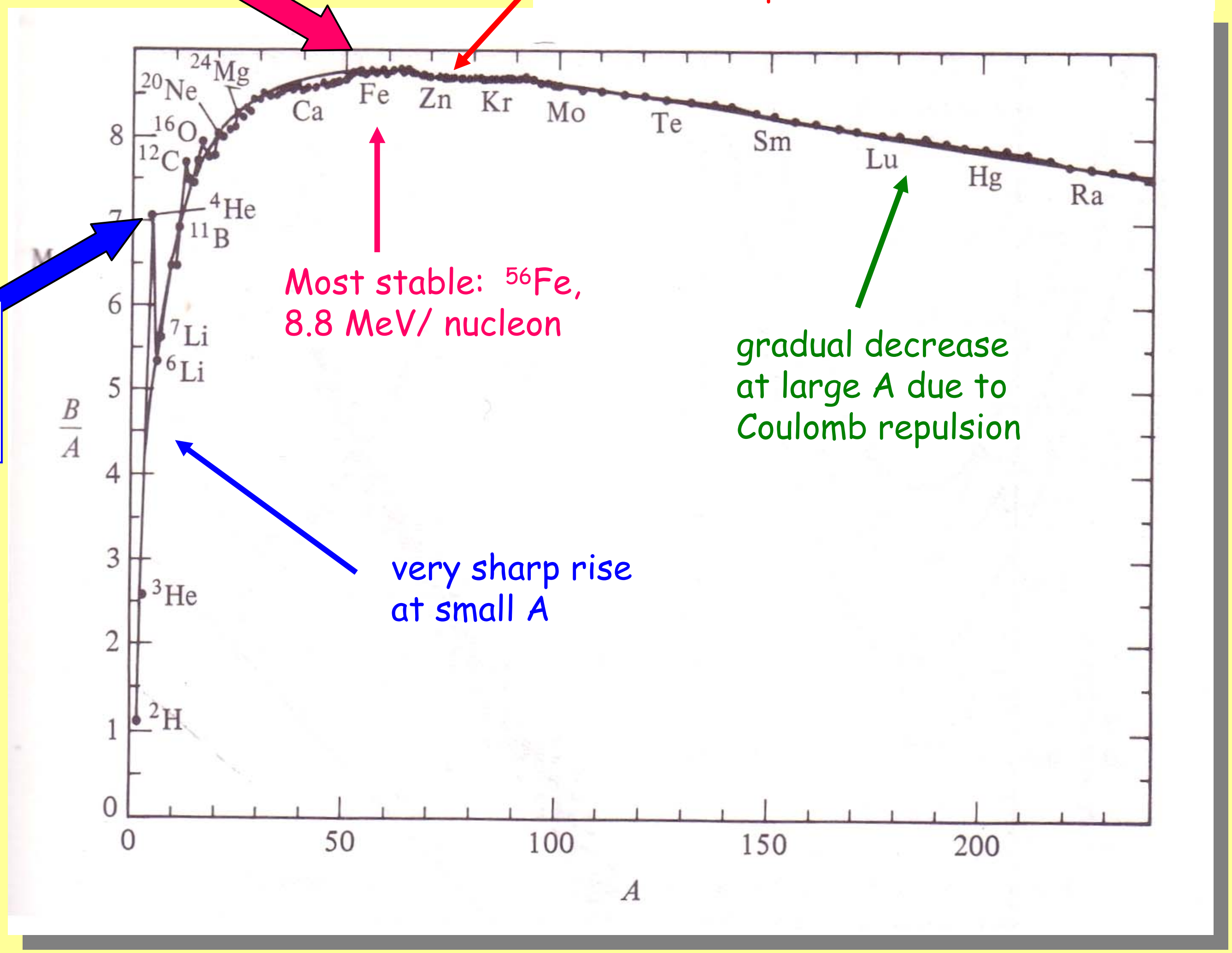


$a_p = 34$ MeV (not shown)

One more look at the Binding Energy per Nucleon curve:

Solid line: fit to the semi-empirical formula

almost flat, apart from Coulomb effects



some large oscillations at small mass

Most stable: ^{56}Fe, 8.8 MeV/ nucleon

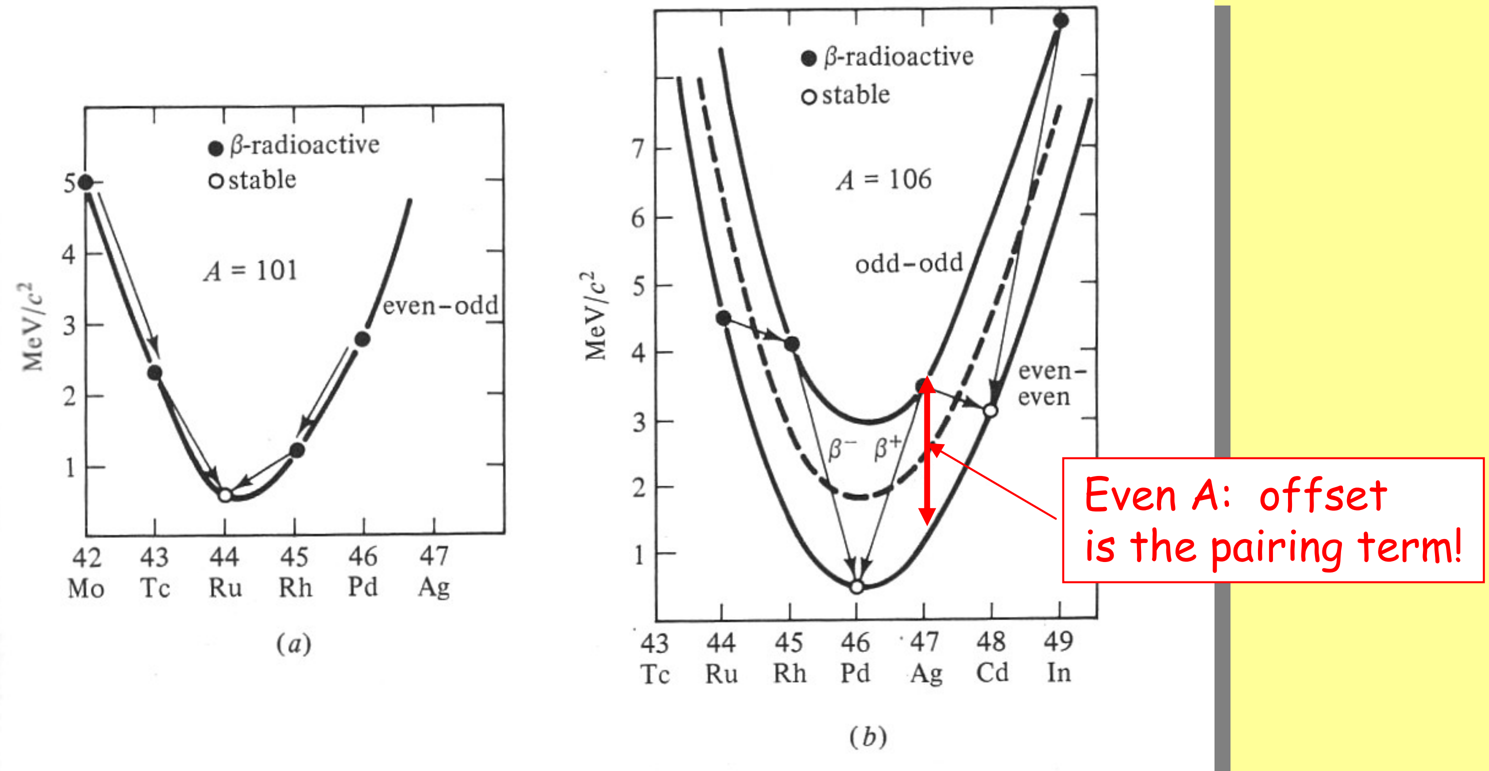
gradual decrease at large A due to Coulomb repulsion

very sharp rise at small A

Semi-Empirical Mass (binding energy) Formula (SEMF) implications:

$$B(Z,A) \;=\; a_V A - a_S A^{2/3} - a_C\, Z\,(Z-1)\, A^{-1/3} \;-\; a_A\,(A-2Z)^2\, A^{-1} \;+\; \delta$$

Stable nuclei have the maximum B for a given A; for constant mass number, B is quadratic in Z → "mass parabolas", e.g.:



2.6. Variation of mass with Z for (a) odd-A isobars ($A = 101$); (b) even-A isobars ($A = 106$). (From Segrè, E., *Nuclei and Particles.* Benjamin (1977).)

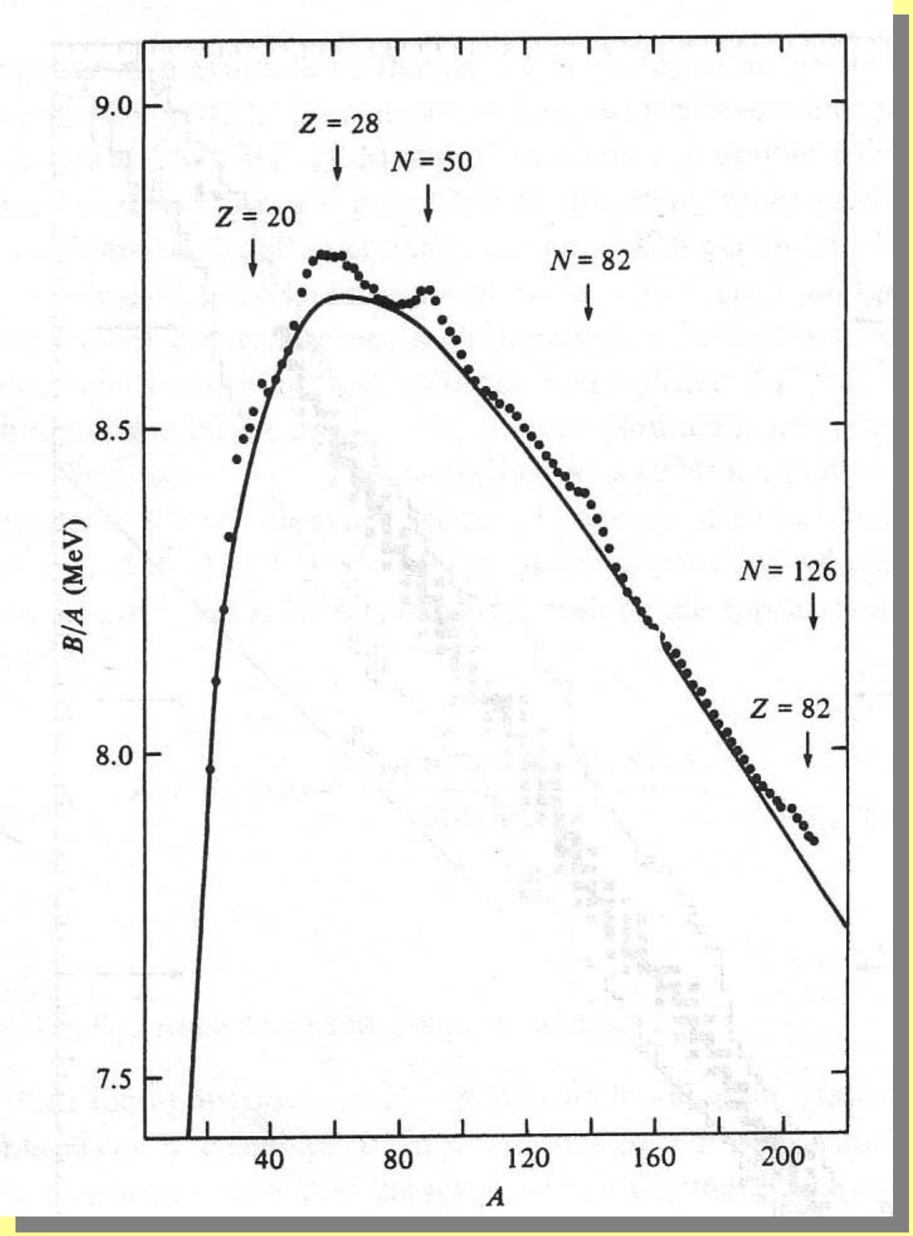
We already noted that there were some marked deviations from the SEMF curve at small mass number, e.g. A = 4.

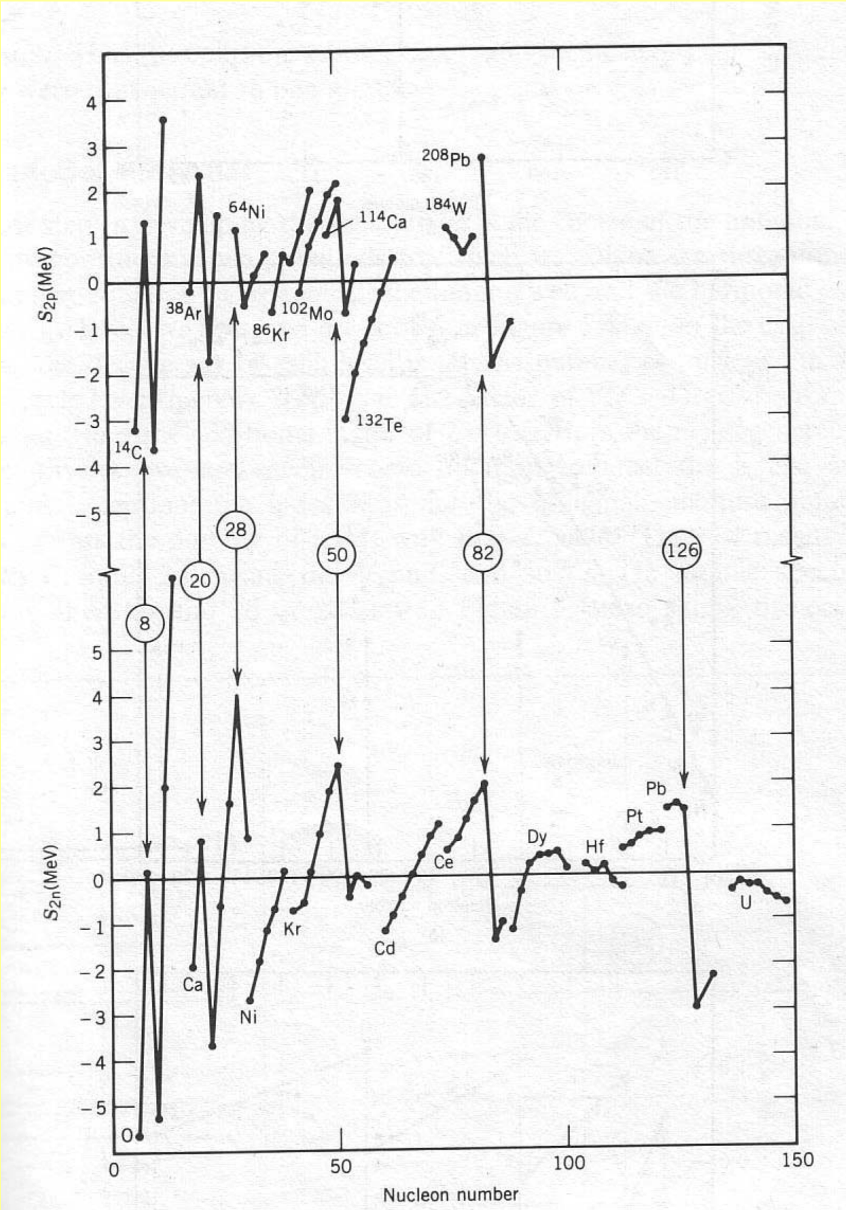
On an enlarged scale, a systematic pattern of deviations occurs, with maxima in B occurring for certain "magic" values of N and Z, given by:

N/Z = 2, 8, 20, 28, 50, 82, 126

These values of neutron and proton number are anomalously stable with respect to the average – the pattern must therefore reflect something important about the average nuclear potential V(r) that the neutrons and protons are bound in....

(NB, the most stable nucleus of all is ^{56}Fe, which has Z = 28, N = 28, "double magic" ...)





- Energy required to remove a pair of neutrons or protons from a given nucleus is referred to as S_{2n} or S_{2p}
- like the ionization energy for atoms, but the pairing force is so strong in nuclei that systematics are more easily seen comparing nuclei that differ by 2 nucleons
- the same pattern of "magic numbers" appears – large separation energies correspond to particularly stable nuclei:

N/Z = 2, 8, 20, 28, 50, 82, 126 ....

Systematics are reminiscent of the periodic structure of atoms, which results from filling independent single-particle electron states with electrons in the most efficient way consistent with the Pauli principle, but the magic numbers are different:

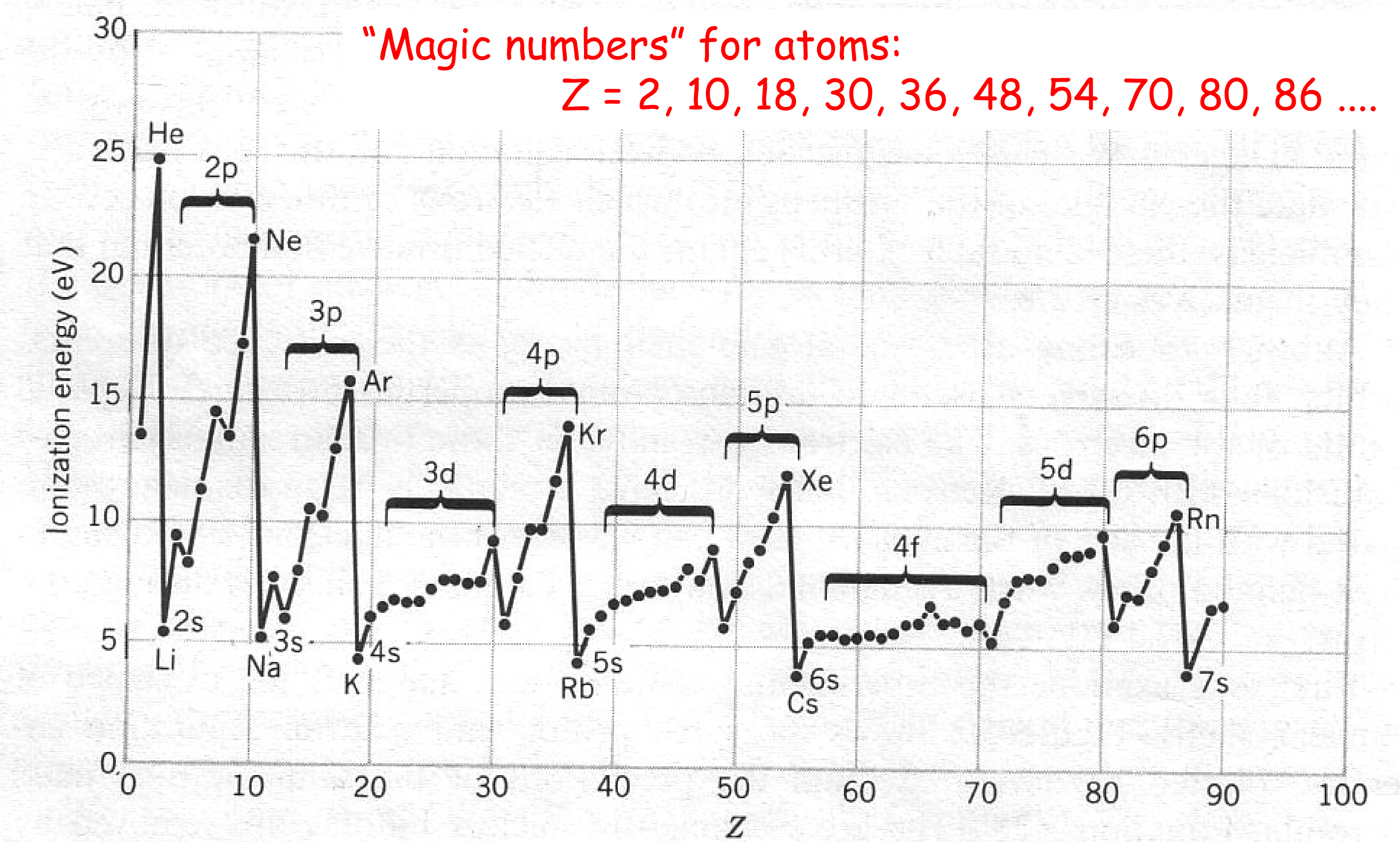
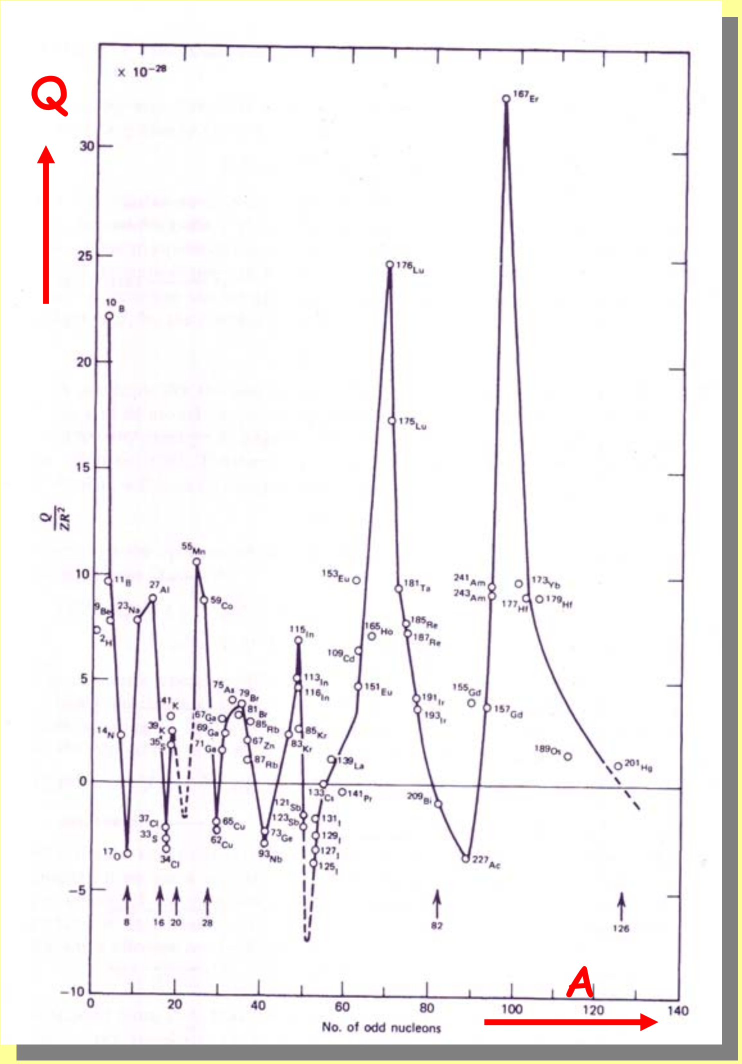


Figure 5.1 Atomic radius (top) and ionization energy (bottom) of the elements. The smooth variations in these properties correspond to the gradual filling of an atomic shell, and the sudden jumps show transitions to the next shell.

Self-consistent approximation: assume the quantum state of the i^{th} nucleon can be found by solving a Schrödinger equation for its interaction via an average nuclear potential $V_N(r)$ due to the other (A-1) nucleons:

$$\left[\frac{-\hbar^2}{2\mu} \nabla^2 + V_N(r) \right] \psi_{nlm}(\vec{r}_i) = E_{nl}\, \psi_{nlm}(\vec{r}_i)$$



Assume a spherically symmetric potential $V_N(r)$; then the eigenstates have definite orbital angular momentum, and the standard radial and angular momentum quantum numbers *(n,l,m)* as indicated.

(Justification: measured quadrupole moments of nuclei are relatively small, at least near the "magic numbers" that we are interested in explaining; midway between the last two magic numbers, ie around Z or N = 70, 100, the picture changes, and we will have to use a different approach, but at least for the lighter nuclei this assumption should be reasonable.)

If we choose the right potential function $V_N(r)$, then the wave function for the whole nucleus can be written as a product of the single particle wave functions for all *A* nucleons, or at least schematically:

$$\Psi_{Nucleus}(\vec{r}) = \prod_{i=1}^{A} \psi_{nlm}(\vec{r}_i)$$

oversimplification here... actually, it has to be written as an antisymmetrized product wavefunction since the nucleons are identical Fermions – the procedure is well-documented in advanced textbooks in any case!

With total angular momentum given by:

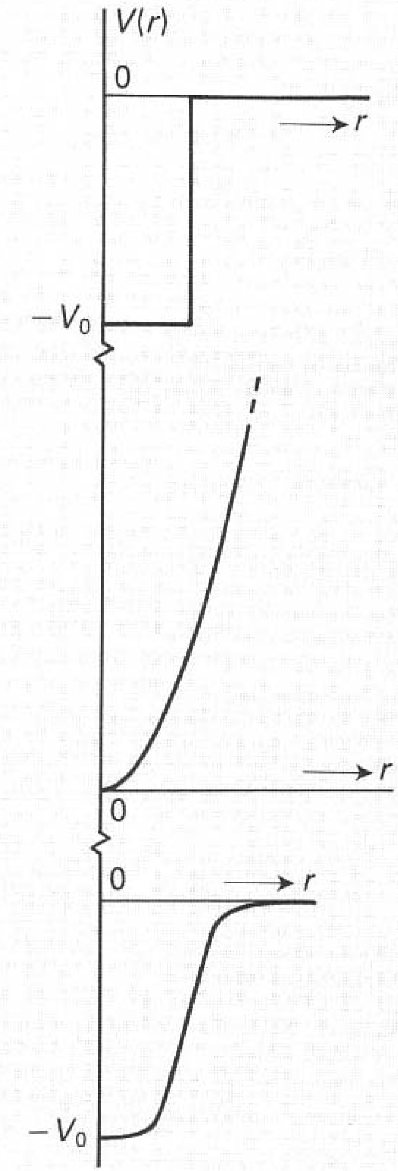
$$\vec{J} = \sum_{i=1}^{A} \vec{j}_i \,, \quad \vec{j}_i = \vec{\ell}_i + \vec{s}_i \,, \quad (s = \frac{1}{2})$$

And parity:

$$\pi = \prod_{i=1}^{A} (-1)^{\ell_i}$$

Always + for an even number of nucleons...

What to use for $V_N(r)$? – three candidate potential functions:



Square well

$r<R$: $V(r) = -V_0$

$r>R$: $V(r) = 0$

Advantage: easy to write down
Disadvantages:
- numerical solutions only
- edges unrealistically sharp

Harmonic oscillator

$$V(r) = \tfrac{1}{2} M\omega^2 r^2$$

Advantage: easy to write down and can be solved analytically
Disadvantage: potential should not go to infinity, have to cut off the function at some finite r and adjust parameters to fit data.

Saxon–Woods

$$V(r) = \frac{-V_0}{1+\exp\left(\frac{r-R}{d}\right)}$$

Advantage: same shape as measured charge density distributions of nuclei. smooth edge makes sense

Disadvantage:
numerical solution needed

- since both potentials are spherically symmetric, the only difference is in the radial dependence of the wave functions
- amazingly, when parameters are adjusted to make the average potential the same, as shown in the top panel, there is remarkably little difference in the radial probability densities for these two potential energy functions!
- this being the case, the simplicity of the harmonic oscillator potential means that it is strongly preferred as a model for nuclei

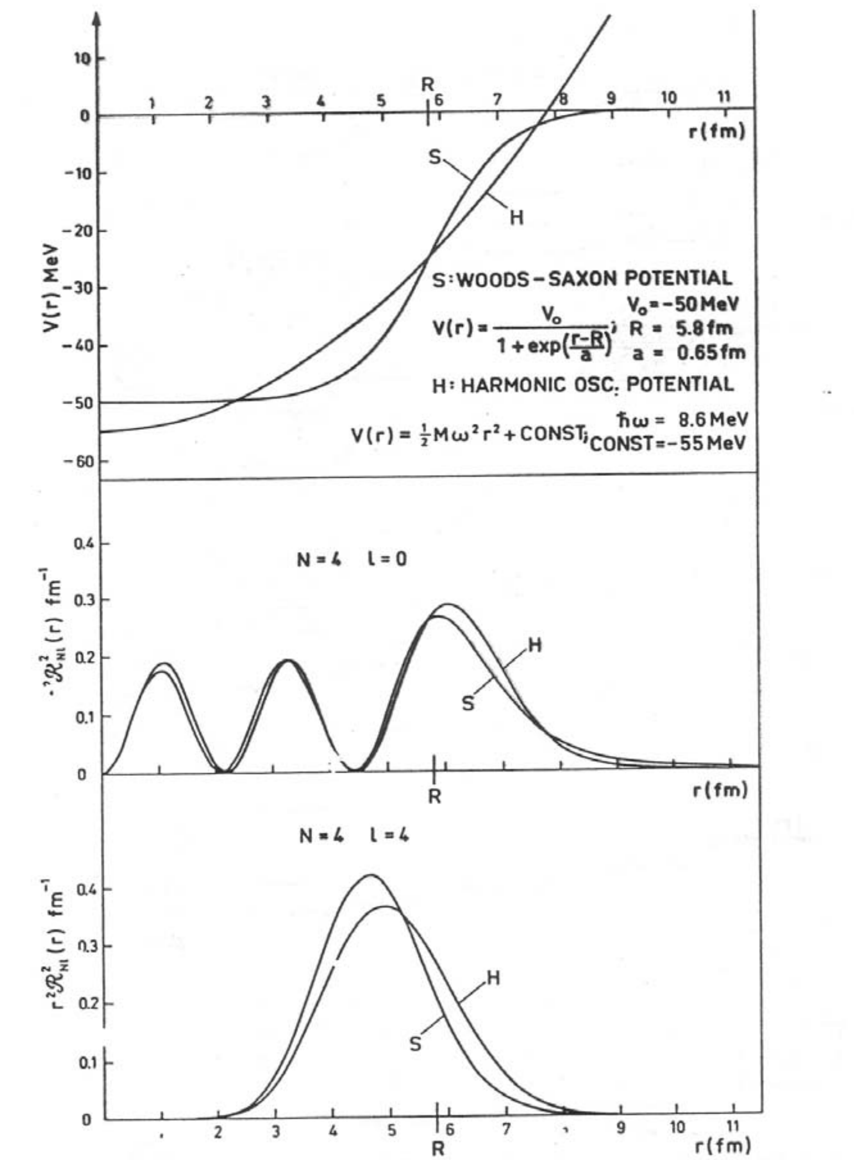


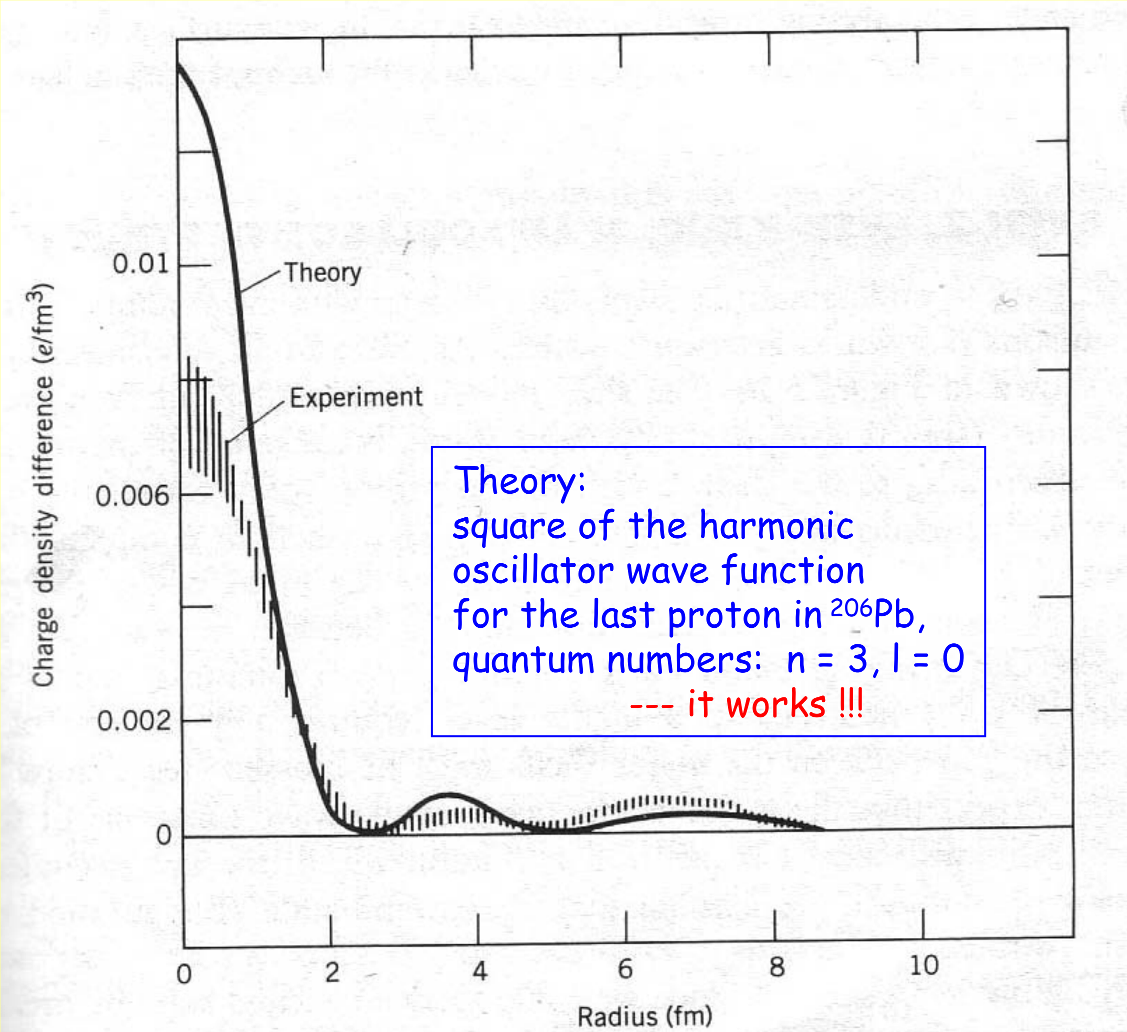
Figure 2-22 The square of the wave function times r^2 for the harmonic oscillator and the Woods-Saxon potential are plotted in units of fm^{-1}.

Evidence that this works:

electric charge density, measured via electron scattering:

$$\rho(\vec{r}) = e \sum_{i=1}^{Z} |\psi_i(\vec{r})|^2$$

charge density difference between ^{205}Tl and ^{206}Pb is proportional to the square of the wave function for the extra proton in ^{206}Pb, i.e. we can actually measure the square of the wave function for a single proton in a complex nucleus this way!



Theory:
square of the harmonic oscillator wave function for the last proton in ^{206}Pb, quantum numbers: $n = 3$, $l = 0$
--- it works !!!

But, we still have a problem explaining the magic numbers – next class!